5月8日

H９　神戸

1．$A=\begin{pmatrix} a & a & a \\ b & b & b \\ c & c & c \end{pmatrix}$ に対して

（１）$A^2=kA$となるkを求めよ。
（２）A^nを求めよ。
（３）$(E+A)^n$を求めよ。

2．$f(x)=\log(1+x)$に対して

（１）$f'(x)$，$f''(x)$，$f'''(x)$を求めよ。
（２）第３次までマクローリン展開せよ。
（３）$\lim_{x\to\infty}\left\{x-x^2\log\left(1+\dfrac{1}{x}\right)\right\}$を求めよ。

3．$f(x)=\begin{cases} \sin x & \left(0\le x\le\dfrac{\pi}{2}\right) \\ 0 & \left(x<0, \dfrac{\pi}{2}<x\right) \end{cases}$ に対して

（１）$\int_0^{\frac{\pi}{2}}\sin x\,dx$，$\int_0^{\frac{\pi}{2}}x\sin x\,dx$，$\int_0^{\frac{\pi}{2}}x^2\sin x\,dx$　をそれぞれ求めよ。
（２）$f(x)$を確立変数Xの密度関数とする。
平均$E(X)$，分散$V(X)$を求めよ。

4．$\displaystyle\iint_D\frac{dx\,dy}{\sqrt{1-x^2-y^2}}$　　$D=\{(x,y)\mid x^2+y^2\le x\}$
を求めよ。

H9 神戸

1、 $A=\begin{pmatrix} a & a & a \\ b & b & b \\ c & c & c \end{pmatrix}$

(1) $A^2 = kA$, k?

$$A^2 = \begin{pmatrix} a & a & a \\ b & b & b \\ c & c & c \end{pmatrix}\begin{pmatrix} a & a & a \\ b & b & b \\ c & c & c \end{pmatrix} = \begin{pmatrix} a(a+b+c) & a(a+b+c) & a(a+b+c) \\ b(a+b+c) & b(a+b+c) & b(a+b+c) \\ c(a+b+c) & c(a+b+c) & c(a+b+c) \end{pmatrix}$$

$$= \underbrace{(a+b+c)}_{\substack{\parallel \\ k}} \underbrace{\begin{pmatrix} a & a & a \\ b & b & b \\ c & c & c \end{pmatrix}}_{\substack{\parallel \\ A}} = kA$$

(2) $A^2 = kA$, $A^3 = kA^2 = k^2A$, …… $A^n = k^{n-1}A = (a+b+c)^{n-1}A$

(3) $(E+A)^n = \sum_{k=0}^{n} {}_nC_k E^{n-k} A^k = \sum_{k=0}^{n} {}_nC_k A^k$

(∵ 二項定理)

$$= \sum_{k=0}^{n} {}_nC_k (a+b+c)^{k-1} A$$

$$= \frac{A}{a+b+c} \sum_{k=0}^{n} {}_nC_k (a+b+c)^k \cdot 1^{n-k}$$

$$= \frac{A}{a+b+c} (1+a+b+c)^n$$

$$= \frac{(1+a+b+c)^n}{a+b+c} A$$

2、 $f(x)=\log(1+x)$

(1) $f'(x)=\dfrac{1}{1+x}=(1+x)^{-1}$

$f''(x)=-\dfrac{1}{(1+x)^2}=-(1+x)^{-2}$

$f'''(x)=2\dfrac{1}{(1+x)^3}=2(1+x)^{-3}$

マクローリン展開

$f(x)=f(0)+f'(0)x+\cdots\cdots$

$+\dfrac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}+\dfrac{f^{(n)}(\theta x)}{n!}x^n$

$(0\leqq\theta<1)$

(2) $f(0)=\log 1=0$, $f'(0)=1$, $f''(0)=-1$

$\therefore \log(1+x)=x-\dfrac{1}{2}x^2+\dfrac{2}{3!\,(1+\theta x)^3}x^3$ $\underline{(0\leqq\theta<1)}$

(3) (2)より) $\log(1+x)=x-\dfrac{1}{2}x^2+\dfrac{1}{3}\dfrac{x^3}{(1+\theta x)^3}$

$\therefore \log\left(1+\dfrac{1}{x}\right)=\dfrac{1}{x}-\dfrac{1}{2}\dfrac{1}{x^2}+\dfrac{1}{3}\dfrac{1/x^3}{(1+\theta/x)^3}$

$=\dfrac{1}{x}-\dfrac{1}{2x^2}+\dfrac{1}{3x^3(1+\theta/x)^3}$

$x-x^2\log\left(1+\dfrac{1}{x}\right)=x-x^2\left\{\dfrac{1}{x}-\dfrac{1}{2x^2}+\dfrac{1}{3x^3(1+\theta/x)^3}\right\}$

$=\cancel{x-x}+\dfrac{1}{2}-\dfrac{1}{3x(1+\theta/x)^3}$

$\xrightarrow[(x\to\infty)]{}\dfrac{1}{2}$

Date 5・15・後 No. 3

$$\therefore \lim_{x\to\infty}\left\{x - x^2\log\left(1+\frac{1}{x}\right)\right\} = \frac{1}{2}$$

参： $f^{(n)}(x) = (-1)^{n-1}(n-1)!\,(1+x)^{-n}$

$$\frac{f^{(n)}(0)}{n!} = (-1)^{n-1}\frac{(n-1)!}{n!} = (-1)^{n-1}\frac{1}{n}$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots\cdots$$

$f(x)$：確率変数 X の密度関数

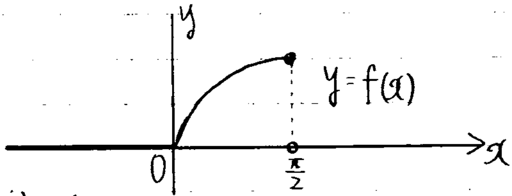
$0 \leqq f(x) \qquad \int_{-\infty}^{\infty} f(x)\,dx = 1$

$P(a<X<b) = \int_a^b f(x)\,dx$

(平均) $\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

(分散) $\sigma^2 = V(X) = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

3. $f(x) = \begin{cases} \sin x & (0 \leqq x \leqq \frac{\pi}{2}) \\ 0 & (x<0,\ \frac{\pi}{2}<x) \end{cases}$



(1) $\int_0^{\pi/2} \sin x\,dx = [-\cos x]_0^{\frac{\pi}{2}} = 0-(-1) = 1$ //

$\int_0^{\frac{\pi}{2}} x\sin x\,dx = -\int_0^{\frac{\pi}{2}} x(\cos x)'dx = -\left\{[x\cos x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\cos x\,dx\right\}$

$= -\left\{0 - [\sin x]_0^{\frac{\pi}{2}}\right\} = 1-0 = 1$ //

$$\int_0^{\pi/2} x^2 \sin x \, dx$$

$$= -\int_0^{\frac{\pi}{2}} x^2 (\cos x)' dx = -\left\{ [x^2 \cos x]_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} x \cos x \, dx \right\}$$

$$= 2\int_0^{\frac{\pi}{2}} x (\sin x)' dx = 2\left\{ [x \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, dx \right\}$$

$$= 2\left\{ \frac{\pi}{2} + [\cos x]_0^{\frac{\pi}{2}} \right\} = 2\left\{ \frac{\pi}{2} - 1 \right\} = \pi - 2 \,/\!/$$

(2) $$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx = \left(\int_{-\infty}^{0} + \int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\infty} \right) x f(x) dx$$

$$= \int_0^{\frac{\pi}{2}} x \sin x \, dx = 1$$

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$$

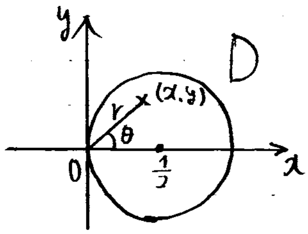
$$= \int_0^{\frac{\pi}{2}} x^2 \sin x \, dx - 1^2 = \pi - 2 - 1 = \pi - 3 \,/\!/$$

4. $$I = \iint_D \frac{dx\,dy}{\sqrt{1-x^2-y^2}}$$ $$D = \{ (x, y) \mid x^2 + y^2 \leqq x \}$$

$$\downarrow$$

$$x^2 - x + y^2 \leqq 0$$

$$\left(x - \frac{1}{2}\right)^2 + y^2 \leqq \left(\frac{1}{2}\right)^2$$

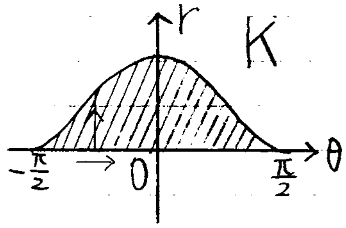


$$\frac{(r\cos\theta)^2 + (r\sin\theta)^2}{r^2} \leqq r\cos\theta$$

$$\uparrow \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ dx\,dy = r\,dr\,d\theta \end{cases}$$

$$\therefore r \leqq \cos\theta$$

$$\left(-\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2} \right)$$



$$I = \iint_D \frac{dx\,dy}{\sqrt{1-x^2-y^2}} = \iint_K \frac{r\,dr\,d\theta}{\sqrt{1-r^2}}$$

(極座標変換)

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_0^{\cos\theta} \frac{r}{\sqrt{1-r^2}}\, dr \right\} d\theta$$

$$\left[\begin{array}{ll} t = \sqrt{1-r^2} & \searrow \therefore \displaystyle\int \frac{r\,dr}{\sqrt{1-r^2}} = \int \frac{-t\,dt}{t} = -\int dt = -t \\ t^2 = 1 - r^2 & \qquad = -\sqrt{1-r^2} \\ 2t\,dt = -2r\,dr & \\ \therefore r\,dr = -t\,dt \nearrow & \end{array}\right]$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{\left[-\sqrt{1-r^2}\right]_0^{\cos\theta}}_{\theta\text{の偶}} d\theta = 2\int_0^{\frac{\pi}{2}} \left[-\sqrt{1-r^2}\right]_0^{\cos\theta} d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left\{-\sqrt{1-\cos^2\theta} - (-1)\right\} d\theta = 2\int_0^{\frac{\pi}{2}} 1 - \sin\theta\, d\theta$$

$$= 2\left\{ [\theta]_0^{\frac{\pi}{2}} + [\cos\theta]_0^{\frac{\pi}{2}} \right\} = 2\left(\frac{\pi}{2} + 0 - 1\right) = \pi - 2 \,//$$